

A new infinite family of Cameron-Liebler line classes

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Abstract

We construct a new infinite family of Cameron-Liebler line classes in $\text{PG}(3, q)$ with parameter $x = \frac{q^2+1}{2}$ for all odd q .

1 Introduction

Let $\text{PG}(3, q)$ denote the 3-dimensional projective space over the finite field \mathbb{F}_q . For a set \mathcal{L} of lines in $\text{PG}(3, q)$, let $\overline{\mathcal{L}}$ denote the complementary set of lines. A *spread* of $\text{PG}(3, q)$ is a set of $q^2 + 1$ lines that partition the set of points.

We say that \mathcal{L} is a *Cameron-Liebler line class* with parameter x in $\text{PG}(3, q)$, if there exists a non-negative integer x such that, for every spread S of $\text{PG}(3, q)$, one has:

$$|S \cap \mathcal{L}| = x.$$

It can be seen from the definition that $\overline{\mathcal{L}}$ is then a Cameron-Liebler line class with parameter $q^2 + 1 - x$, so that we may assume $x \leq \frac{q^2+1}{2}$. An empty set of lines ($x = 0$), the set of all lines in a plane ($x = 1$) or, dually, through a point ($x = 1$) are trivial examples of Cameron-Liebler line classes. If the point is not in the plane, then the union of the previous two examples with $x = 1$ gives a slightly less trivial Cameron-Liebler line class with parameter $x = 2$.

Cameron-Liebler line classes first appeared in the study [3] (and were given their name in [11]) of collineation groups of $\text{PG}(n, q)$, $n \geq 3$, that have equally many orbits on lines and on points. Under the Klein correspondence, Cameron-Liebler line classes are translated to tight sets of the Klein quadric being, thus, a special case of a tight set of a polar space (see [4, 1]). For more comprehensive background on this topic, we refer to the recent papers [6, 8, 10, 7, 1].

It was conjectured in [3] that the only Cameron-Liebler line classes are the examples mentioned above, i.e., $x \leq 2$. The first counterexample was found by Drudge [5] in $\text{PG}(3, 3)$ with $x = 5$, which was generalised later by Bruen and Drudge [2] to an infinite family having parameter $x = \frac{q^2+1}{2}$ for all odd q . The first counterexample in characteristic 2 was found in [9]. With the aid of computer and using some clever ideas about possible symmetries of Cameron-Liebler line classes, Rodgers [13] constructed many more new examples for certain x and prime powers q . Very recently, some of them have been shown in [1], [6] to be a part of a new infinite family of Cameron-Liebler line classes with parameter $x = \frac{q^2+1}{2}$ for $q \equiv 5$ or $9 \pmod{12}$. (In fact, a line class of the family found in [1], [6] has parameter $\frac{q^2-1}{2}$, however, it is disjoint with a plane, which is a Cameron-Liebler line class with parameter 1, so that the union of their lines is a Cameron-Liebler line class with parameter $\frac{q^2-1}{2} + 1$.)

In this note, we first describe a switching-like operation in Cameron-Liebler line classes that satisfy some necessary conditions (see Lemma 2.1). We then show in Lemma 2.3 that these conditions may only hold for line classes with $x = q^2$ or $x = \frac{q^2+1}{2}$. Applying this switching operation to the line classes found by Bruen and Drudge, we construct another infinite family of Cameron-Liebler line classes in $\text{PG}(3, q)$ with parameter $x = \frac{q^2+1}{2}$ for all odd q , and show that they are not equivalent to the line classes of Bruen and Drudge, unless $q = 3$ (see Theorem 3.3).

2 Switching in Cameron-Liebler line classes

For a point P and a plane π of $\text{PG}(3, q)$, let $\text{Star}(P)$ and $\text{Line}(\pi)$ denote the set of all lines on P or in π , respectively.

Lemma 2.1 *Let \mathcal{L} be a Cameron-Liebler line class such that there exists an incident point-plane pair (P, π) satisfying the following conditions:*

- (1) $(\text{Line}(\pi) \setminus \text{Star}(P)) \cap \mathcal{L} = \emptyset$,
- (2) $\text{Star}(P) \setminus \text{Line}(\pi) \subseteq \mathcal{L}$.

Then

$$\mathcal{L}' := \mathcal{L} \cup (\text{Line}(\pi) \setminus \text{Star}(P)) \setminus (\text{Star}(P) \setminus \text{Line}(\pi))$$

is a Cameron-Liebler line class with the same parameter.

Proof: For any spread S of $\text{PG}(3, q)$ we have that S contains either a line of $\text{Star}(P) \cap \text{Line}(\pi)$, or a line $\ell \in \text{Line}(\pi) \setminus \text{Star}(P)$ and a line $m \in \text{Star}(P) \setminus \text{Line}(\pi)$. In the former case, $S \cap \mathcal{L} = S \cap \mathcal{L}'$, while in the latter case $S \cap \mathcal{L}' = (S \cap \mathcal{L}) \cup \{m\} \setminus \{\ell\}$. Thus, $|S \cap \mathcal{L}'| = |S \cap \mathcal{L}|$ holds in both cases, and so \mathcal{L}' is a Cameron-Liebler line class. \blacksquare

Let \mathcal{L} be a Cameron-Liebler line class, and ℓ a line of $\text{PG}(3, q)$. Then ℓ lies in $q + 1$ planes π_1, \dots, π_{q+1} and contains $q + 1$ points P_1, \dots, P_{q+1} . Define the square matrix $T(\ell) = (t_{ij})$ of size $q + 1$ with integer entries given by

$$t_{ij} := |((\text{Line}(\pi_i) \cap \text{Star}(P_j)) \setminus \{\ell\}) \cap \mathcal{L}|, \quad 1 \leq i, j \leq q + 1.$$

The set consisting of the matrix T , and every matrix obtained from this one by a permutation of the rows and a permutation of the columns is called the *pattern* of ℓ with respect to \mathcal{L} . We represent each pattern by one of its matrices. This concept was introduced in [8], where the following result has been proved.

Proposition 2.2 *Let \mathcal{L} be a Cameron-Liebler line class with parameter x , let ℓ be a line of $\text{PG}(3, q)$, and $T = (t_{ij})$ the pattern of ℓ .*

(a) *For any $i \in \{1, \dots, q + 1\}$*

$$\sum_{j=1}^{q+1} t_{ij} = |\text{Line}(\pi_i) \cap \mathcal{L} \setminus \{\ell\}| \quad \text{and} \quad \sum_{j=1}^{q+1} t_{ji} = |\text{Star}(P_i) \cap \mathcal{L} \setminus \{\ell\}|.$$

(b) *For all $k, l \in \{1, \dots, q + 1\}$*

$$\sum_{i=1}^{q+1} t_{il} + \sum_{j=1}^{q+1} t_{kj} = \begin{cases} x + (q + 1)t_{kl} & \text{if } \ell \notin \mathcal{L} \\ x + (q + 1)(t_{kl} + 1) - 2 & \text{if } \ell \in \mathcal{L}. \end{cases}$$

(c) *$t_{kl} + t_{rs} = t_{ks} + t_{rl}$ for all $k, l, r, s \in \{1, \dots, q + 1\}$.*

(d)

$$\sum_{i,j=1}^{q+1} t_{ij}^2 = \begin{cases} x(q + x) & \text{if } \ell \notin \mathcal{L} \\ q^3 + q^2 + (x - 1)^2 + q(x - 1) & \text{if } \ell \in \mathcal{L}. \end{cases}$$

Lemma 2.3 *Let \mathcal{L} be a Cameron-Liebler line class such that there exists an incident point-plane pair (P, π) satisfying the conditions of Lemma 2.1. Then the parameter x of \mathcal{L} is equal to q^2 or $\frac{q^2+1}{2}$.*

Proof: Up to taking the complement to a line set and the point-plane duality in $\text{PG}(3, q)$, we may assume that there exists a line ℓ of $\text{Star}(P) \cap \text{Line}(\pi) \setminus \mathcal{L}$. Let T be the pattern of ℓ such that, without loss of generality, its first row corresponds to π , and its first column corresponds to P . Then the conditions of Lemma 2.1 imply that $t := t_{11} = |\text{Star}(P) \cap \text{Line}(\pi) \cap \mathcal{L}|$, and $t_{1,j} = q$ and $t_{j,1} = 0$ for all $j \in \{2, \dots, q+1\}$. By Proposition 2.2 (c), we see that $t_{ij} = q - t_{11}$ for all $i, j \in \{2, \dots, q+1\}$.

Further, Proposition 2.2 (b) applied to the first row and column of T , and Proposition 2.2 (d) applied to the pattern T give the following equations:

$$\begin{cases} t + q^2 + t = x + t(q+1), \\ t^2 + q^3 + q^2(q-t)^2 = x(q+x), \end{cases},$$

which yield $t = 0$ and $x = q^2$ (and thus \mathcal{L} is the complement to a Cameron-Liebler line class with parameter 1), or $t = \frac{q+1}{2}$ and $x = \frac{q^2+1}{2}$. \blacksquare

3 Application of switching

From Lemma 2.3 we see that the only non-trivial case, where the switching operation of Lemma 2.1 may be applied, is the case $x = \frac{q^2+1}{2}$. There exist at least two infinite families of Cameron-Liebler line classes with parameter $x = \frac{q^2+1}{2}$, namely, the first counterexamples to the Cameron-Liebler conjecture constructed by Bruen and Drudge in [2] and the line classes recently found in [1] and independently in [6]. Fortunately, the former satisfies the conditions of Lemma 2.1 (while the latter do not), and applying the switching operation indeed produces a new Cameron-Liebler line class, not equivalent to the original one, if $q > 3$. In this section we give the necessary details.

First of all, let us recall the construction by Bruen and Drudge. Let q be an odd prime power, and \mathcal{Q} an elliptic quadric of $\text{PG}(3, q)$ with the corresponding quadratic form Q . The set of $q+1$ tangents \mathcal{T}_P to a point $P \in \mathcal{Q}$ can be divided into two subsets, say $\mathcal{T}_P^1, \mathcal{T}_P^2$, of size $(q+1)/2$ each, depending on whether a tangent line contains a point $P' \neq P$ such that $Q(P')$ is a square in \mathbb{F}_q . Note if $Q(P')$ is a square in \mathbb{F}_q , then all the points on the tangent PP' satisfy this property, as $Q(P + cP') = c^2Q(P')$.

Denote by \mathcal{T}^i the set $\cup_{P \in \mathcal{Q}} \mathcal{T}_P^i$, $i \in \{1, 2\}$. Let \mathcal{S} and \mathcal{E} be the sets of secant and external lines to \mathcal{Q} , respectively. Then any of

$$\mathcal{S} \cup \mathcal{T}^i, \quad \mathcal{E} \cup \mathcal{T}^j, \quad i, j \in \{1, 2\},$$

is a Cameron-Liebler line class of parameter $\frac{q^2+1}{2}$.

Since all these line classes are equivalent under the action of $\text{PTL}(4, q)$ and the polarity induced by \mathcal{Q} (see [4]), we may choose, without loss of generality, \mathcal{L} to be $\mathcal{S} \cup \mathcal{T}^1$. For a point P_1 of \mathcal{Q} and its tangent plane τ_{P_1} , one can see that

$$(\text{Line}(\tau_{P_1}) \setminus \text{Star}(P_1)) \subset \mathcal{E} \subset \overline{\mathcal{L}}, \quad \text{Star}(P_1) \setminus \text{Line}(\tau_{P_1}) \subset \mathcal{S} \subset \mathcal{L},$$

so that (P_1, τ_{P_1}) satisfies the condition of Lemma 2.1, and the line class \mathcal{L}' defined by

$$\mathcal{L}' := \mathcal{L} \cup (\text{Line}(\tau_{P_1}) \setminus \text{Star}(P_1)) \setminus (\text{Star}(P_1) \setminus \text{Line}(\tau_{P_1}))$$

is a Cameron-Liebler line class with parameter $\frac{q^2+1}{2}$.

Our aim now is to show that \mathcal{L}' is not equivalent to \mathcal{L} unless $q = 3$. For $q = 3$, we can either apply Drudge's classification of Cameron-Liebler line classes in $\text{PG}(3, 3)$ [5] that determined that, up to equivalence, there is a unique Cameron-Liebler line class with parameter 5, or it can be checked with the aid of computer that \mathcal{L}' is projectively equivalent to $\overline{\mathcal{L}}$ for this value of q . From now on, we assume that $q > 3$.

Lemma 3.1 *A plane π of $\text{PG}(3, q)$ contains $\frac{q+1}{2}$, or $\frac{q(q+1)}{2}$, or $\frac{(q+1)(q+2)}{2}$ lines of \mathcal{L} .*

Proof: If π is a tangent plane to \mathcal{Q} , then $|\text{Line}(\pi) \cap \mathcal{L}| = \frac{q+1}{2}$ by the construction of \mathcal{L} . Suppose that π is a secant plane so that $\pi \cap \mathcal{Q}$ is a conic. Under the polarity, say ρ , induced by \mathcal{Q} , every tangent line to the conic in π is mapped to a tangent line to \mathcal{Q} on $\rho(\pi)$. Therefore, all tangent lines to the conic in π are either in \mathcal{T}^1 or in \mathcal{T}^2 . In the former case, π contains $\binom{q+1}{2} + q + 1$ lines from \mathcal{L} , in the latter case $|\text{Line}(\pi) \cap \mathcal{L}| = \binom{q+1}{2}$. ■

Lemma 3.2 *A point P of $\text{PG}(3, q)$ is on $q^2 + \frac{q+1}{2}$, or $\frac{q(q-1)}{2}$, or $\frac{q(q+1)}{2} + 1$ lines of \mathcal{L} .*

Proof: If $P \in \mathcal{Q}$, then $|\text{Star}(P) \cap \mathcal{L}| = \frac{q+1}{2} + q^2$ by the construction of \mathcal{L} . Suppose that $P \notin \mathcal{Q}$. If P is on a tangent line from \mathcal{T}^i for $i \in \{1, 2\}$, then all tangent lines to \mathcal{Q} through P are in \mathcal{T}^i . Let P' be a point of \mathcal{Q} such that PP' is a tangent line to \mathcal{Q} , and consider all secant planes π_1, \dots, π_q containing the line PP' . Recall that every point not on a conic in a projective plane lies on 0 or 2 tangents, see [12, 14]. Since $\pi_i \cap \mathcal{Q}$ is a conic, and PP' is a tangent line to the conic, we conclude that P lies on 2 tangents and $\frac{q-1}{2}$ secants to $\pi_i \cap \mathcal{Q}$. Thus, $|\text{Star}(P) \cap \mathcal{L}| = \frac{q(q-1)}{2}$, if $PP' \in \mathcal{T}^2$, or $|\text{Star}(P) \cap \mathcal{L}| = \frac{q(q-1)}{2} + q + 1$, if $PP' \in \mathcal{T}^1$. ■

Theorem 3.3 *The line classes \mathcal{L} and \mathcal{L}' are not equivalent under the action of $\text{PTL}(4, q)$ or a duality.*

Proof: Following the notation from the above, one can see that the plane τ_{P_1} contains $\frac{q+1}{2} + q^2$ lines of \mathcal{L}' . Since, for a point $P_2 \in \mathcal{Q}$, $P_2 \neq P_1$, one has $\tau_{P_1} \cap \tau_{P_2} \in \mathcal{E}$, the plane τ_{P_2} contains $\frac{q+1}{2} + 1$ lines of \mathcal{L}' . It now follows from Lemmas 3.1, 3.2 that the intersection numbers of \mathcal{L}' with respect to planes and points of $\text{PG}(3, q)$ are different from those of \mathcal{L} or $\overline{\mathcal{L}}$. ■

We also note that \mathcal{L}' is not equivalent to a line class of the family found in [1], [6], since there is no plane (or, dually, a point with all lines on it) contained in or disjoint from \mathcal{L}' . In particular, in $\text{PG}(3, 5)$, there exist at least three pairwise non-equivalent Cameron-Liebler line classes with $x = \frac{q^2+1}{2} = 13$ (namely, the example by Bruen and Drudge, its switched mate by Theorem 3.3, and the example found in [6] and [1]). In fact, up to equivalence, these are the only Cameron-Liebler line classes with given x in $\text{PG}(3, 5)$ (the details will be given elsewhere).

The line class \mathcal{L}' contains only the one incident point-plane pair, namely, (P_1, τ_{P_1}) , satisfying the conditions of Lemma 2.1, and, clearly, switching of \mathcal{L}' with respect to it gives the line class \mathcal{L} . Since, for $q > 3$, there is a unique switched mate for \mathcal{L}' (namely, \mathcal{L}), it follows that its stabiliser $G_{\mathcal{L}'}$ is a subgroup of the stabiliser $G_{\mathcal{L}}$. The stabiliser $G_{\mathcal{L}}$ of a Bruen-Drudge line class is a subgroup of index two of $\text{P}\Gamma\text{O}^-(4, q)$, i.e., the subgroup that fixes \mathcal{T}^1 and \mathcal{T}^2 . Thus, $G_{\mathcal{L}'}$ is the stabiliser of the point P_1 in $G_{\mathcal{L}}$. Then, for $q = p^h$, where p is a prime, $G_{\mathcal{L}'}$ has order $q^2(q^2 - 1)h$, and is isomorphic to $\text{AGL}(1, q^2) \rtimes C_h$.

We expect that the only non-trivial Cameron-Liebler line classes satisfying the conditions of Lemma 2.1 are the examples of Bruen and Drudge and their switched mates.

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